$\boxed{0 \vdash 1}$

logic in color

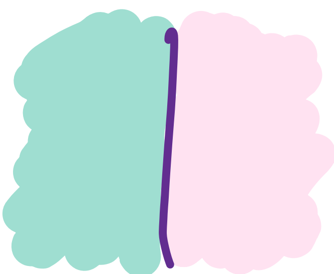
#2 : duality & equipments

A 2-category is [Rel) [data]

- a collection of objects — type

set — A

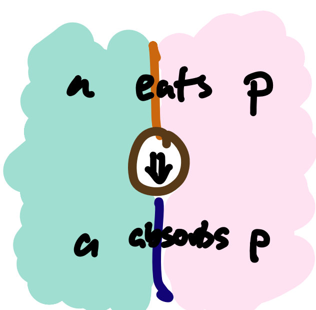
- a collection of morphisms each with source & target object — judgement



relation — $A\ a \xrightarrow{R} b\ B$

$R : A \times B \to \{T, F\}$

- a collection of 2-morphisms each with source & target morphism — inference



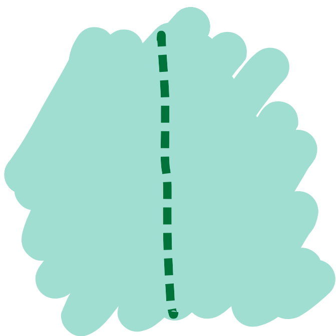
implication

$$\frac{A\ a \xrightarrow{R} b\ B}{A\ a \xrightarrow{S} b\ B}\ \gamma$$

$\forall a \in A, b \in B.$
$R[a,b] \Rightarrow S[a,b]$

[note: colors]

- for each object an identity morphism

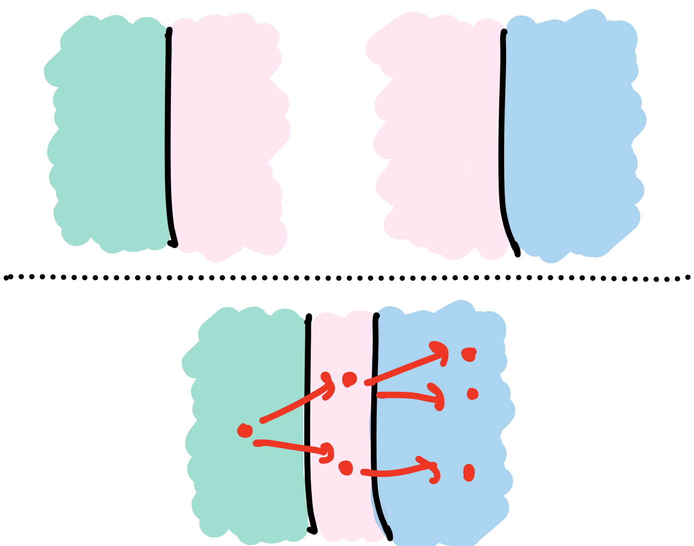


equality relation

$$Aa \vdash a'A$$

$$a = a'$$

- on morphisms a composition



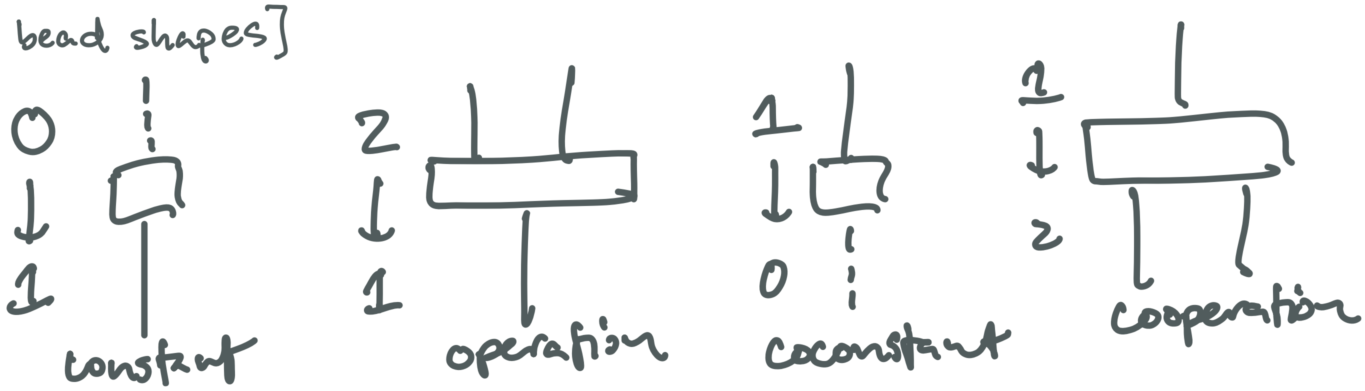
$$Aa \xrightarrow{R} bB \quad Bb \xrightarrow{U} cC$$

$$\underline{Aa \xrightarrow{R \circ U} cC}$$

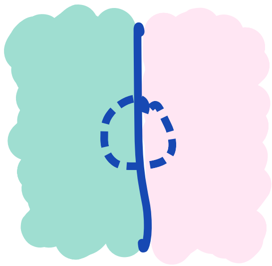
$$= \exists b \in B.\ aRb \wedge bUc$$

which is associative and unital

[note: bead shapes]

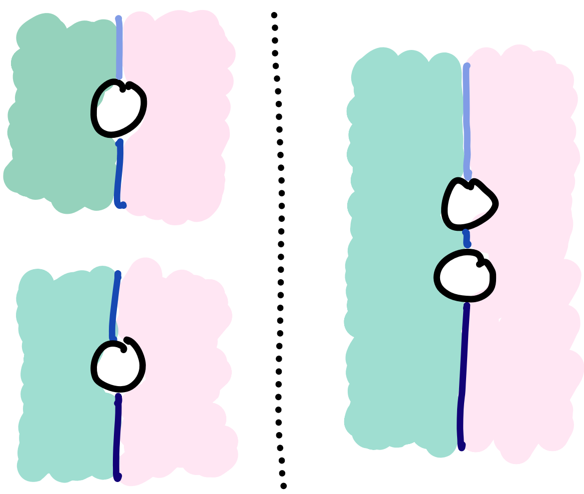


- for each morphism an identity 2-morphism



$$\frac{Aa \xrightarrow{S} bB}{Aa \xrightarrow{S} bB}$$

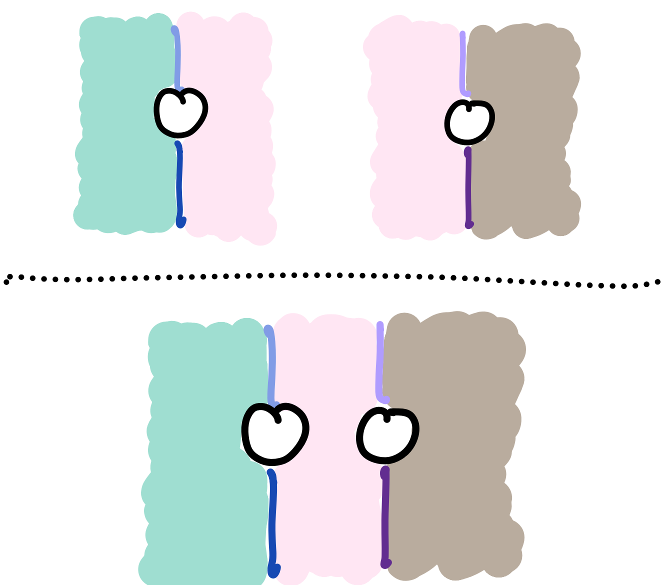
- on 2-morphisms a sequence composition



$$\frac{Aa \xrightarrow{R} bB}{Aa \xrightarrow{S} bB}\,\gamma \qquad \frac{Aa \xrightarrow{S} bB}{Aa \xrightarrow{T} bB}\,\delta$$

$$\frac{Aa \xrightarrow{R} bB}{Aa \xrightarrow{T} bB}\,\gamma\delta$$

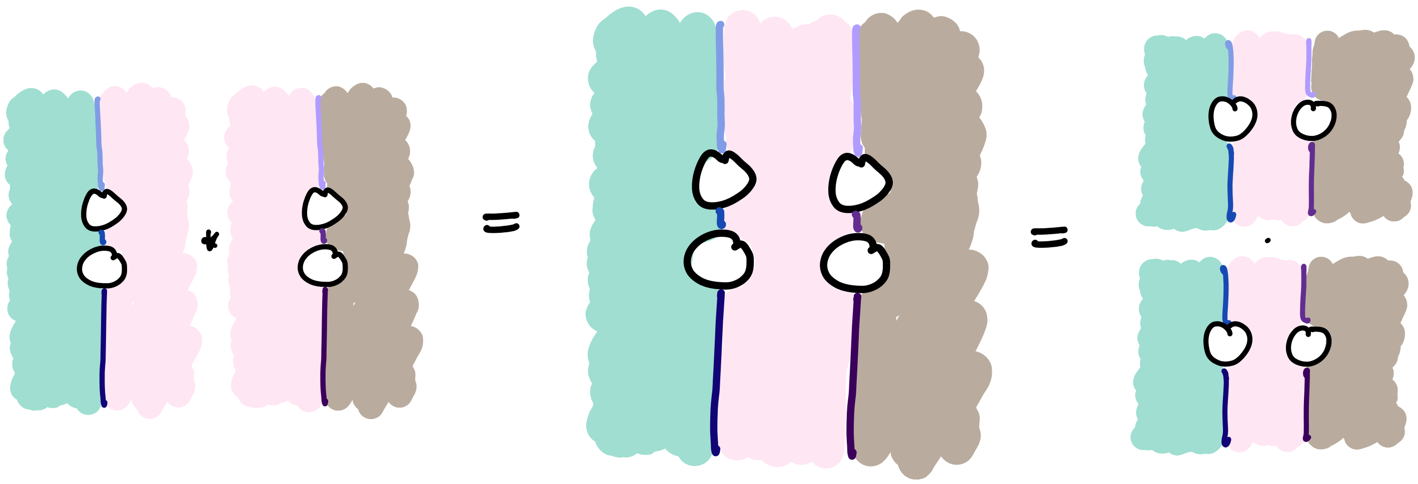
- on 2-morphisms a parallel composition



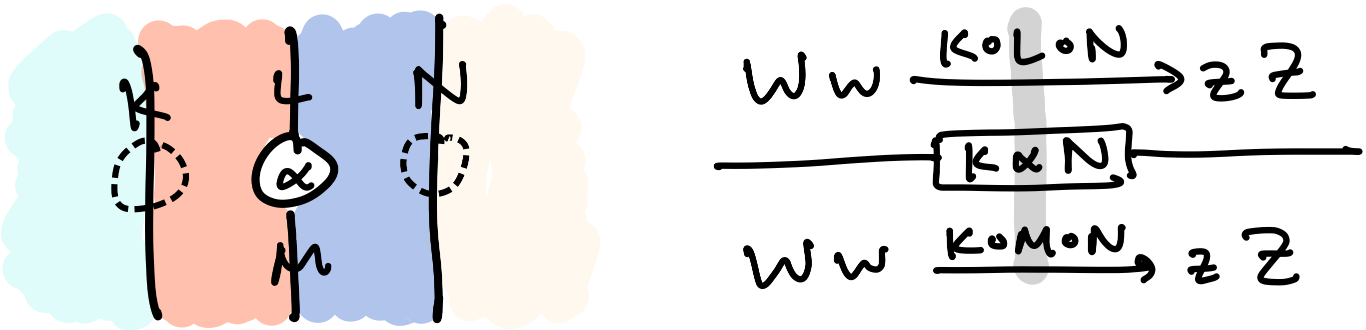
$$\frac{Aa \xrightarrow{R} bB}{Aa \xrightarrow{S} bB}\,\gamma \qquad \frac{Bb \xrightarrow{U} cC}{Bb \xrightarrow{V} cC}\,\varepsilon$$

$$\frac{Aa \xrightarrow{R \circ U} cC}{Aa \xrightarrow{S \circ V} cC}\,\gamma * \varepsilon$$

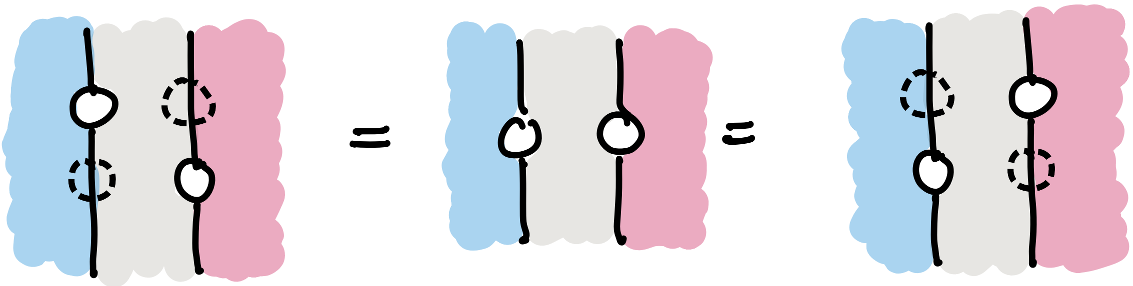
- sequence & parallel composition are associative & unital, and compatible :



note: identity beads allow for "whiskering"

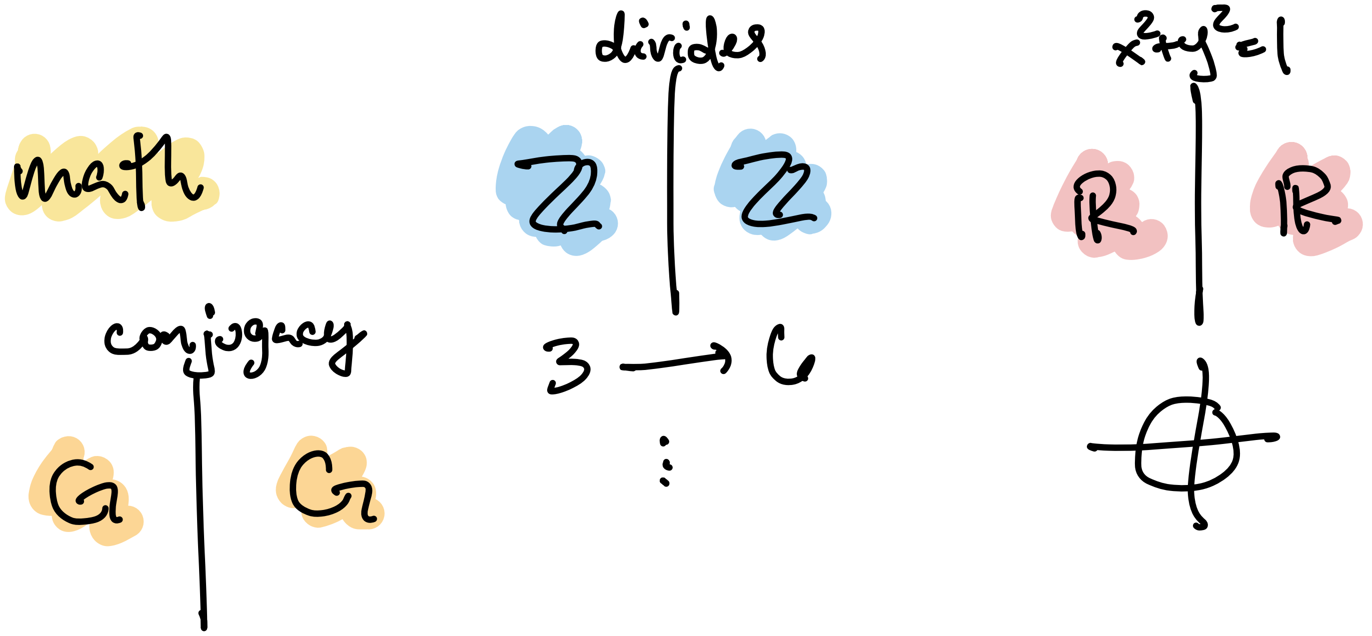


and compatibility implies that parallel beads can slide past each other



examples

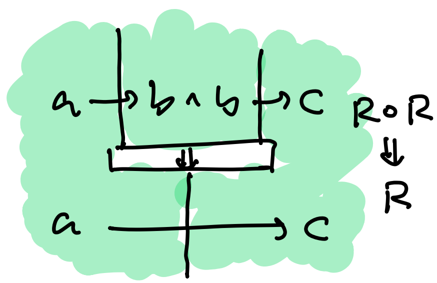
math



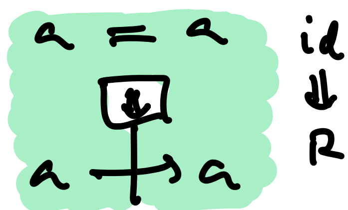
$g\,C\,g' = \exists h \in G.\ hgh^{-1} = g'$

How to draw

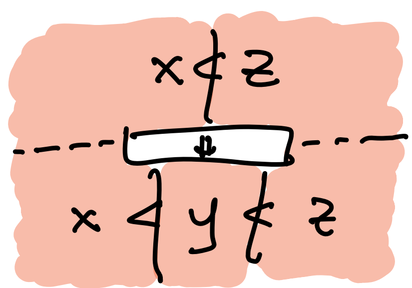
- transitive



- reflexive

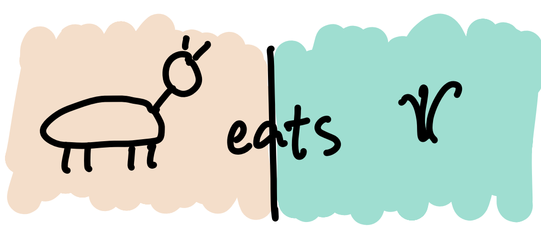


- dense



duality

Every relation has a converse



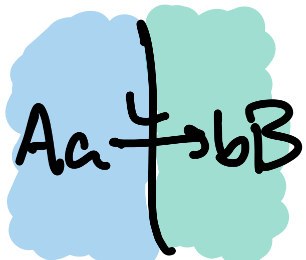
which is not necessarily an inverse.

However, in 2 dimensions a pair of strings can be

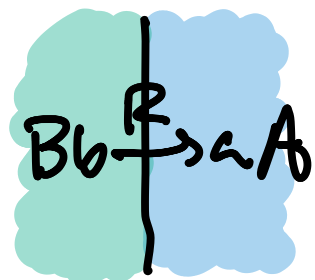
$A \rightleftarrows B$

/different/ dual equivalent

An adjunction is a pair

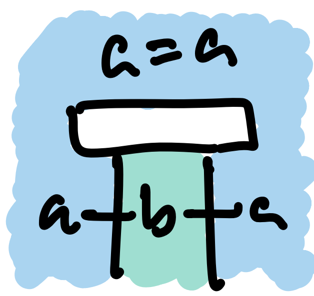


left adjoint



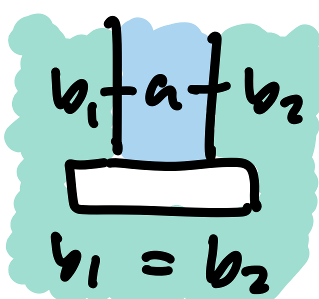
right adjoint

with a unit



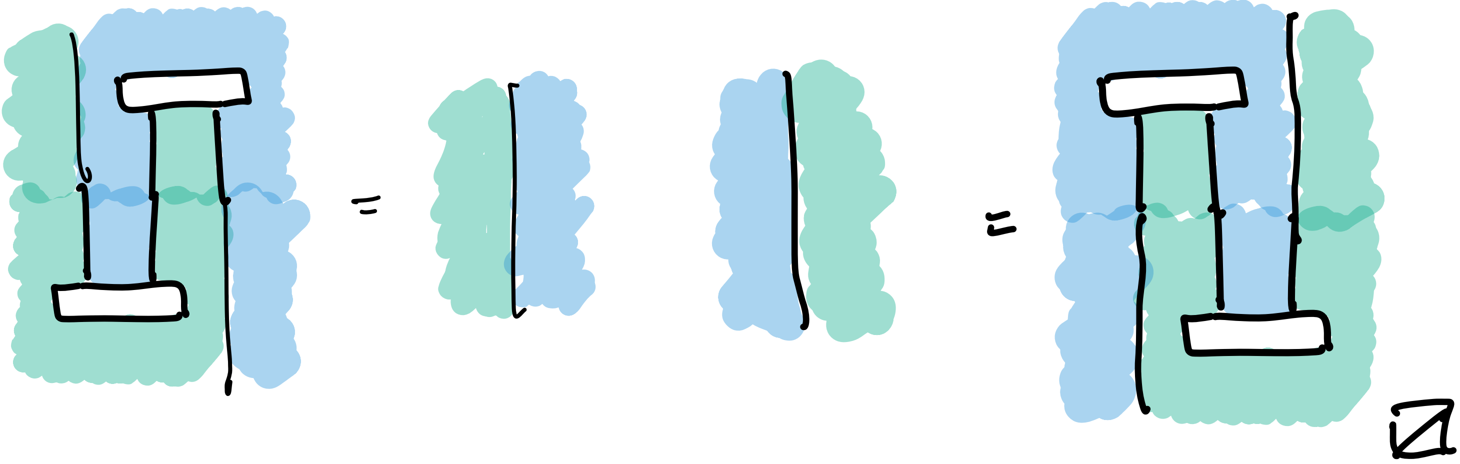
$$\frac{Aa \vdash a'A}{Aa \xrightarrow{LoR} a'A}\ \eta$$

+ counit



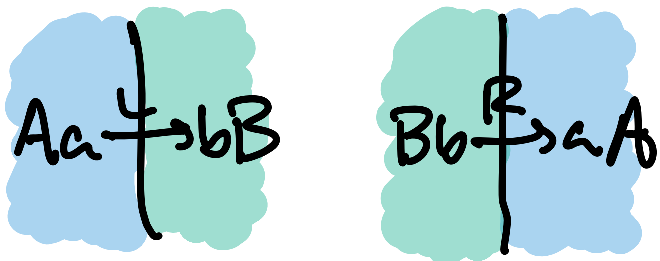
$$\frac{Bb \xrightarrow{RoL} b'B}{Bb \vdash b'B}\ \varepsilon$$

which cancel along each string:

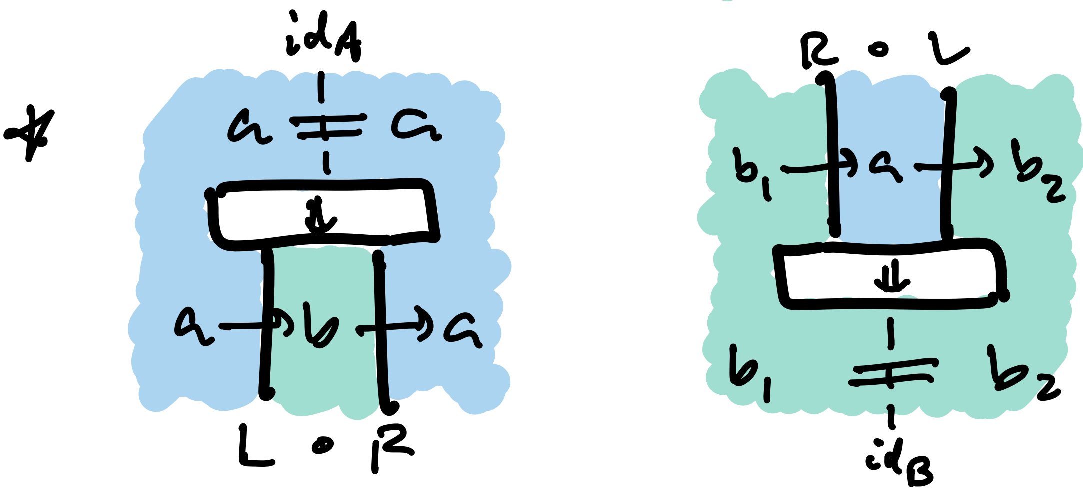


so each string can "bend + unbend" – this basic geometry goes a <u>long</u> way.

* what does it mean in <u>Rel</u>?



*

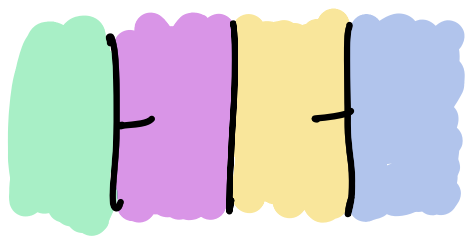


total
"∀ input ∃ output"

deterministic
"output is unique"

so, L is a <u>function</u>! + R is a cofunction.

note:

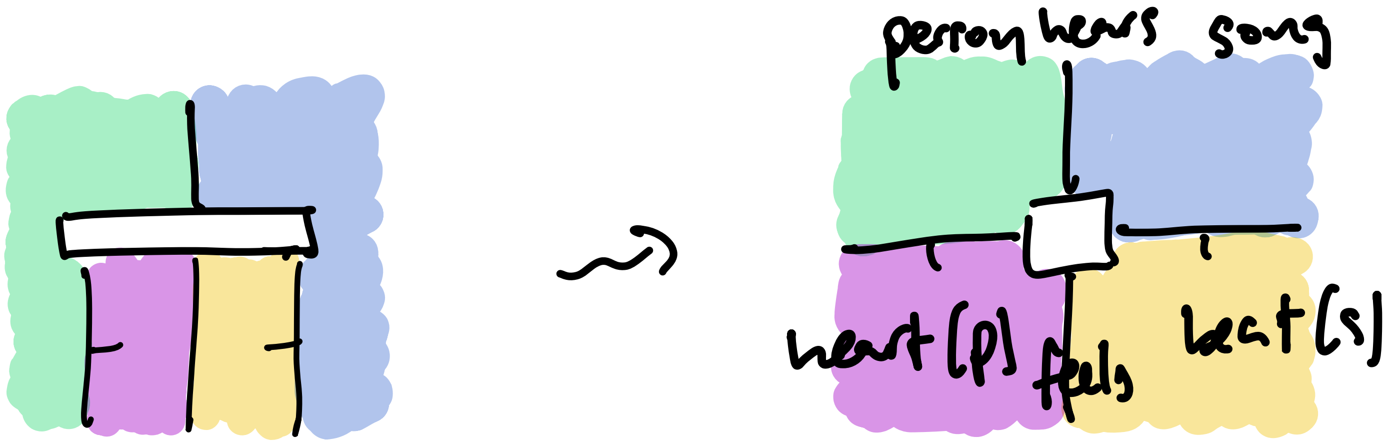


substitution

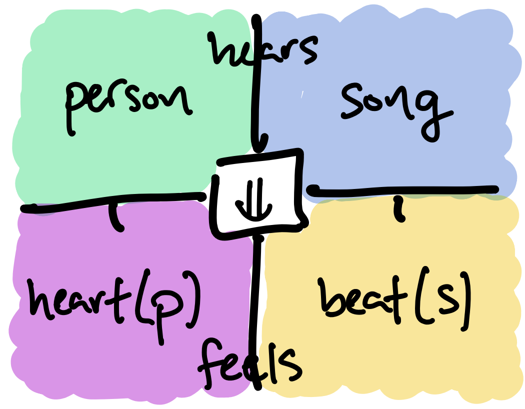
$$a \xrightarrow{f \circ R \circ g^{-1}} b \quad = \quad f(a) \xrightarrow{R} g(b)$$

in reality, we want to think about
both functions & relations

so they each deserve
★ their own dimension ★



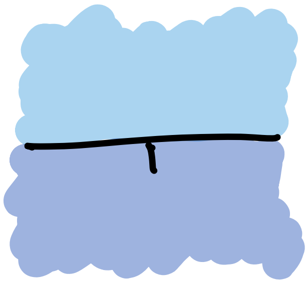
these inferences
are more natural.
for example,



A double category is...

like a 2-category, plus

- vertical morphisms



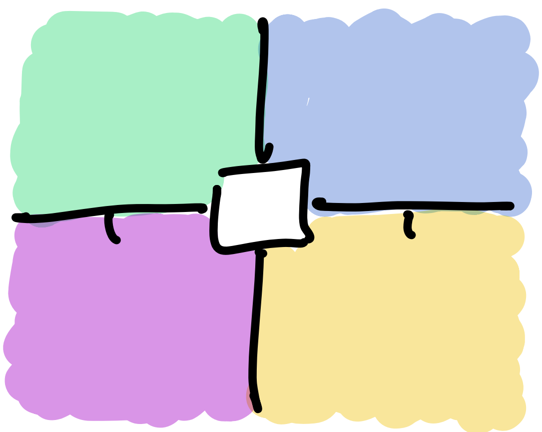
function

term

$$\frac{A\,a \vdash a'\,A}{B\,fa \vdash fa'\,B}\;(f)$$

- vertical composition

- 2-morphisms are squares
with (horizontal) source/target and vertical s/t.



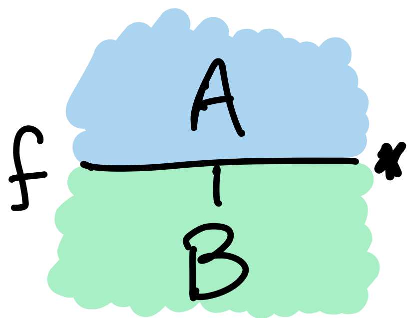
$$\frac{A\,a \xrightarrow{R} b\,B}{X\,f(a) \xrightarrow{S} g(b)\,Y}\;(r)$$

- (composition along both)

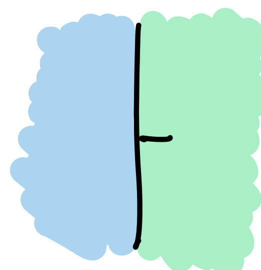
how to formalize that functions can bend into relations :

A fibrant double category ("equipment") is a double category with:

- for each vertical morphism (term) a pair of horizontal morphisms (judgements)

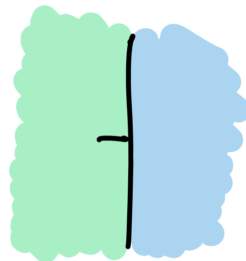


$\rightsquigarrow$



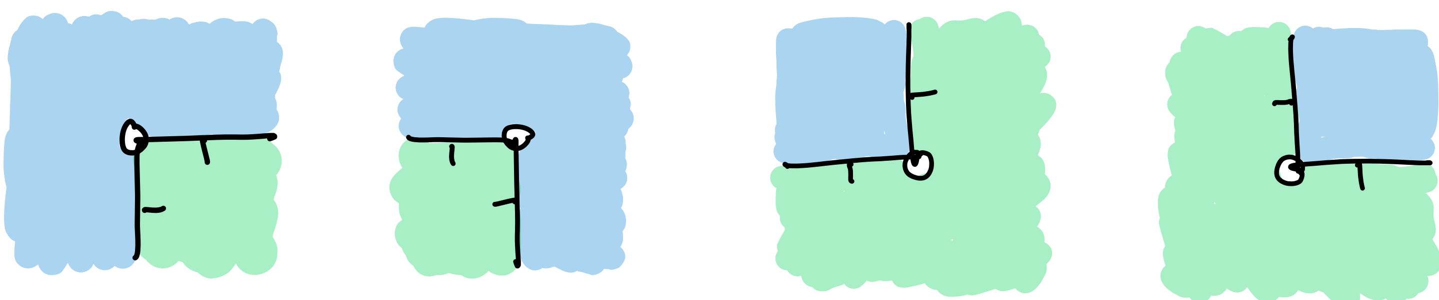
$A\ f(a) \vdash b\ B$

(the first is "left adjoint" to the second)

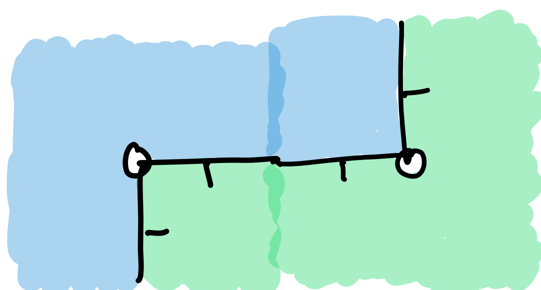


$B\ b \vdash f(a)\ A$

- equipped with 2-morphisms (what do these inferences mean?)



- such that



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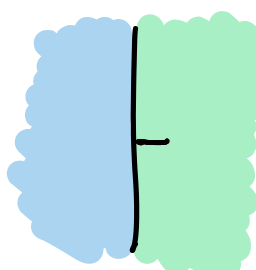

etc.

Puzzles

* what can be expressed so far?
  try some favorite concepts/theorems.
* what more structure do we need?

[Rel has all higher-order logic,
when its structure is expounded.
in two lessons, we'll explore quantifiers.]

[* clearly, there's not enough time
to explore everything in depth.
if you're interested, just email me
at cwill041@ucr.edu]

Thanks!